Differential Calculi and Linear Connections

Aristophanes Dimakis
Institut für Theoretische Physik
Bunsenstraße 9, D-37073 Göttingen

J. Madore
Laboratoire de Physique Théorique et Hautes Energies*
Université de Paris-Sud, Bât. 211, F-91405 Orsay

January, 1996

Abstract

A method is proposed for defining an arbitrary number of differential calculi over a given noncommutative associative algebra. As an example the generalized quantum plane is studied. It is found that there is a strong correlation, but not a one-to-one correspondence, between the module structure of the 1-forms and the metric torsion-free connections on it. In the commutative limit the connection remains as a shadow of the algebraic structure of the 1-forms.

LPTHE Orsay 95/79

*Laboratoire associé au CNRS, URA D0063

1 Introduction

We propose a method of defining an arbitrary number of differential calculi over a given noncommutative associative algebra. We shall especially be interested in the generalized quantum plane, an algebra which has a commutative limit which one can identify with the algebra of rational functions on the 2-plane minus the axes. We shall see that the commutation relations between the elements of the algebra and the 1-forms determine a set of torsion-free metric connections which remain non-trivial in the commutative limit. It is to be expected that the converse is true, that every torsion-free metric connection on the 2-plane determines a set of commutation relations. This would mean that in particular the covariant calculus of Wess & Zumino (1990) is determined by a geometry on the 2-plane.

The differential calculi we introduce here are based on derivations and are similar to those introduced by Dubois-Violette (1988) and Dubois-Violette *et al.* (1989) to construct differential calculi over matrix algebras. We refer to Madore (1995a,b) for a detailed description of how to use this method to construct a sequence of differential calculi over a given matrix algebra. In this previous work the set of derivations were chosen to form the Lie algebra of the special linear group SL_m. With this restriction the number of differential calculi which can be put on a matrix algebra of dimension n is equal to the number of integers m such that the SL_m has an irreducible representation on a space of dimension n. There are always of course at least two, $m = 2$ and $m = n$ but for large n there are in general many more.

In Section 2 we present a general method for constructing differential calculi, based on a set of derivations which do not necessarily close to form a Lie algebra. In Sections 3 and 4 we present some examples using as algebra the generalized quantum plane. In Section 5 we investigate linear connections and show how they depend on the choice of differential calculus as well as, of course, the algebra. By $\mathcal{A}$ we designate an arbitrary associative algebra with unit element and with center $\mathcal{Z}(\mathcal{A})$

2 General Formalism

Of the many differential calculi which can be constructed over an algebra $\mathcal{A}$ the largest is the differential envelope or universal differential calculus $(\Omega_u^*(\mathcal{A}), d_u)$. Every other differential calculus can be considered as a quotient of it. For the definitions we refer, for example, to the book by Connes (1994). Let $(\Omega^*(\mathcal{A}), d)$ be another differential calculus over $\mathcal{A}$. Then there exists a unique d_u-homomorphism ϕ

$$
\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{d_u} & \Omega_u^1(\mathcal{A}) & \xrightarrow{d_u} & \Omega_u^2(\mathcal{A}) & \xrightarrow{d_u} & \cdots \\
\| & & \phi_1 \downarrow & & \phi_2 \downarrow & & \\
\mathcal{A} & \xrightarrow{d} & \Omega^1(\mathcal{A}) & \xrightarrow{d} & \Omega^2(\mathcal{A}) & \xrightarrow{d} & \cdots
\end{array}
\tag{2.1}
$$

of $\Omega_u^*(\mathcal{A})$ onto $\Omega^*(\mathcal{A})$. It is given by

$$\phi(d_u a) = da. \tag{2.2}$$

The restriction ϕ_p of ϕ to each Ω_u^p is defined by

$$\phi_p(a_0 d_u a_1 \cdots d_u a_p) = a_0 da_1 \cdots da_p.$$

Consider a given algebra $\mathcal{A}$ and suppose that we know how to construct an $\mathcal{A}$-module $\Omega^1(\mathcal{A})$ and an application

$$\mathcal{A} \xrightarrow{d} \Omega^1(\mathcal{A}). \tag{2.3}$$

Then using (2.1) there is a method of constructing $\Omega^p(\mathcal{A})$ for $p \geq 2$ as well as the extension of the differential. Since we know $\Omega^1_u(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ we can suppose that ϕ_1 is given. We must construct $\Omega^2(\mathcal{A})$. The simplest consistent choice would be to set

$$\Omega^2(\mathcal{A}) = \Omega^2_u(\mathcal{A})/d_u \mathrm{Ker}\phi_1. \tag{2.4}$$

This is the largest differential calculus consistent with the constraints on $\Omega^1(\mathcal{A})$. The map ϕ_2 is defined to be the projection of $\Omega^2_u(\mathcal{A})$ onto $\Omega^2(\mathcal{A})$ so defined and d is defined by $d(f dg) = df dg$. This procedure can be continued by iteration to arbitrary order in p. See, for example, Baehr *et al.* (1995).

To initiate the above construction we shall define the 1-forms using a set of derivations. We shall suppose that they are interior and exclude therefore the case where $\mathcal{A}$ is commutative. For each integer n let λ_a be a set of n linearly independent elements of $\mathcal{A}$ and introduce the derivations $e_a = \mathrm{ad}\,\lambda_a$. In general the e_a do not form a Lie algebra but they do however satisfy commutation relations as a consequence of the commutation relations of $\mathcal{A}$. Define the map (2.3) by

$$df(e_a) = e_a\, f. \tag{2.5}$$

We shall suppose that there exists a set of n elements θ^a of $\Omega^1(\mathcal{A})$ such that

$$\theta^a(e_b) = \delta^a_b. \tag{2.6}$$

In the examples which we consider we shall show that the θ^a exist by explicit construction. We shall refer to the set of θ^a as a frame or Stehbein. It commutes with the elements $f \in \mathcal{A}$,

$$f\theta^a = \theta^a f. \tag{2.7}$$

The construction of the θ^a from the derivations did not really use the fact that they were inner. For example if the e_a are n linearly independent vector fields on a manifold V of dimension n, that is, n linearly independent outer derivations of the algebra $\mathcal{A} = \mathcal{C}(V)$ of smooth functions on V then $\Omega^*(\mathcal{A})$ is the algebra of de Rham forms.

The $\mathcal{A}$-bimodule $\Omega^1(\mathcal{A})$ is generated by all elements of the form $f dg$ or of the form $(df)g$. Because of the Leibniz rule these conditions are equivalent. Define $\theta = -\lambda_a\theta^a$. Then one sees that

$$df = -[\theta, f] \tag{2.8}$$

and it follows that as a bimodule $\Omega^1(\mathcal{A})$ is generated by one element. It follows also that the 2-form $d\theta + \theta^2$ can be written in the form

$$d\theta + \theta^2 = -\frac{1}{2} K_{ab}\theta^a\theta^b \tag{2.9}$$

with coefficients K_{ab} which lie in $\mathcal{Z}(\mathcal{A})$. By definition

$$f dg(e_a) = f e_a\, g, \qquad (dg) f(e_a) = (e_a\, g) f.$$

Using the frame we can write these as

$$f dg = (f e_a g)\theta^a, \qquad (dg)f = (e_a g) f\theta^a. \tag{2.10}$$

The commutation relations of the algebra constrain the relations between $f dg$ and $(dg)f$ for all f and g.

As a left or right module, $\Omega^1(\mathcal{A})$ is free of rank n. Because of the commutation relations of the algebra or, equivalently, because of the kernel of ϕ_1 in the quotient (2.4) the θ^a satisfy commutation relations. We shall suppose that they can be written in the form

$$\theta^a\theta^b + C^{ab}{}_{cd}\theta^c\theta^d = 0. \tag{2.11}$$

If $C^{ab}{}_{cd} = \delta^a_c\delta^b_d$ then one sees that $\Omega^2(\mathcal{A}) = 0$. It follows from (2.11) that for an arbitrary element f of the algebra

$$[f, C^{ab}{}_{cd}]\theta^c\theta^d = 0.$$

We shall suppose that

$$C^{ab}{}_{ef}C^{ef}{}_{cd} = \delta^a_c\delta^b_d$$

and that the relations (2.11) are complete in the sense that if $A_{ab}\theta^a\theta^b = 0$ we can conclude that

$$A_{ab} - C^{cd}{}_{ab}A_{cd} = 0. \tag{2.12}$$

This will be the case for all the differential calculi which we shall consider on the generalized quantum plane in the next sections. We can conclude then that the $C^{ab}{}_{cd}$ are elements of $\mathcal{Z}(\mathcal{A})$. In ordinary geometry one can choose

$$C^{ab}{}_{cd} = \delta^b_c\delta^a_d$$

and the relation (2.11) expresses the fact that the 1-forms anticommute. Let $\bigwedge^*_C$ be the twisted exterior algebra determined by the the relations (2.11). Then the differential algebra $\Omega^*(\mathcal{A})$ can be factorized as the tensor product

$$\Omega^*(\mathcal{A}) = \mathcal{A} \otimes_{\mathbb{C}} \textstyle\bigwedge^*_C.$$

Because the 2-forms are generated by products of the θ^a one has

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\theta^b\theta^c. \tag{2.13}$$

Without loss of generality we can suppose that the structure elements $C^a{}_{bc}$ satisfy the identities

$$C^a{}_{bc} + C^a{}_{de}C^{de}{}_{bc} = 0. \tag{2.14}$$

It is to be noticed that they do not in general lie $\mathcal{Z}(\mathcal{A})$. In fact from the identity $d(f\theta^a) = d(\theta^a f)$ one sees that

$$\Big(\frac{1}{2}[C^a{}_{bc}, f] + e_{(b} f\delta^a_{c)}\Big)\theta^b\theta^c = 0. \tag{2.15}$$

Using the definition of the derivations one can write this in the form

$$\Big(\frac{1}{2}C^a{}_{bc} + \lambda_{(b}\delta^a_{c)} - \frac{1}{2}D^a{}_{bc}\Big)\theta^b\theta^c = 0 \tag{2.16}$$

with $D^a{}_{bc} \in \mathcal{Z}(\mathcal{A})$. We can suppose that the $D^a{}_{bc}$ satisfy (2.14):

$$D^a{}_{bc} + D^a{}_{de} C^{de}{}_{bc} = 0. \tag{2.17}$$

Using this and the relations (2.11) and (2.14) as well as the completeness assumption (2.12) we can conclude from (2.16) that

$$C^a{}_{bc} - D^a{}_{bc} + \lambda_{(b}\delta^a_{c)} - \lambda_{(d}\delta^a_{e)} C^{de}{}_{bc} = 0. \tag{2.18}$$

The equation (2.16) can also be written in the form

$$d\theta^a = -[\theta, \theta^a] - \frac{1}{2} D^a{}_{bc} \theta^b \theta^c \tag{2.19}$$

with a graded commutator. If $D^a{}_{bc} = 0$ the form (2.8) for the exterior derivative is valid for all elements of $\Omega^*(\mathcal{A})$ and the element θ plays the role of the phase of a generalized Dirac operator in the sense of Connes (1986).

From (2.19) we find that

$$d\theta = -2\theta^2 + \frac{1}{2}\lambda_a D^a{}_{bc} \theta^b \theta^c.$$

Comparing this with (2.9) we conclude that

$$\theta^2 = \frac{1}{2}(\lambda_a D^a{}_{bc} + K_{bc})\theta^b \theta^c. \tag{2.20}$$

If we suppose that K_{bc} satisfies (2.14),

$$K_{ab} + K_{cd} C^{cd}{}_{ab} = 0,$$

then we can conclude that

$$(\lambda_b \lambda_c - C^{de}{}_{bc} \lambda_d \lambda_e - \lambda_a D^a{}_{bc} - K_{bc})\theta^b \theta^c = 0. \tag{2.21}$$

Using again the completeness assumption (2.12) we find

$$\lambda_b \lambda_c - C^{de}{}_{bc} \lambda_d \lambda_e = \lambda_a D^a{}_{bc} + K_{bc}. \tag{2.22}$$

If we introduce the twisted bracket

$$[\lambda_a, \lambda_b]_C = \lambda_a \lambda_b - C^{cd}{}_{ab} \lambda_c \lambda_d.$$

we can rewrite (2.22) in the form

$$[\lambda_b, \lambda_c]_C = \lambda_a D^a{}_{bc} + K_{bc}. \tag{2.23}$$

If we write out in detail the equation $d^2 f = 0$, using (2.12) - (2.14) we find that

$$[e_b, e_c]_C f = e_a f C^a{}_{bc}. \tag{2.24}$$

This is the dual relation to the 'Maurer-Cartan' equation (2.13).

The constraint (2.23) follows from the commutation relations (2.11) we have supposed for the frame as well as from the conditions we have imposed on the coefficients $C^{ab}{}_{cd}$. In the matrix case the general formalism simplifies considerably. The θ^a are given in terms of $d\lambda^a$ by

$$\theta^a = \lambda_b \lambda^a d\lambda^b.$$

The elements of the frame anticommute and one can choose $C^{ab}{}_{cd} = \delta^b_c \delta^a_d$. In Equation (2.19) the first term on the right-hand side vanishes and $D^a{}_{bc} = C^a{}_{bc}$. On the right-hand side of Equations (2.9) and (2.23) we have $K_{ab} = 0$.

3 Calculi based on 2 derivations

Using the construction of the previous section one can construct an infinite sequence of differential calculi over the generalized quantum plane $\mathcal{A}$, the algebra generated by four elements (x, y, x^{-1}, y^{-1}) subject to the relation $xy = qyx$ as well as the usual relations between an element and its inverse. Here q is an arbitrary complex number. The subalgebra generated by (x, y) alone with the covariant differential calculus of Wess & Zumino (1990) is called the quantum plane. In this section we shall consider the case $n = 2$. Define, for $q \neq 1$

$$\lambda_1 = \frac{q}{q-1} y, \qquad \lambda_2 = \frac{q}{q-1} x. \tag{3.1}$$

The normalization has been chosen here so that the structure elements $C^a{}_{12}$ contain no factors q. The λ_a are singular in the limit $q \to 1$ for the same reason as the limit from quantum mechanics:

$$\frac{1}{\hbar} \mathrm{ad}\, p \to \frac{1}{i} \frac{\partial}{\partial q}.$$

We find that

$$e_1 x = -xy, \qquad e_1 y = 0, \qquad e_2 x = 0, \qquad e_2 y = xy. \tag{3.2}$$

These rather unusual derivations are extended to arbitrary polynomials in the generators by the Leibniz rule. From (3.2) we conclude that the commutation relations which follow from (2.10) are

$$\begin{aligned} x dx &= q dx x, \qquad & y dx &= q^{-1} dx y, \\ x dy &= q dy x, \qquad & y dy &= q^{-1} dy y. \end{aligned} \tag{3.3}$$

From these relations if $q \neq -1$ we deduce

$$(dx)^2 = 0, \qquad (dy)^2 = 0, \qquad dx dy + q dy dx = 0. \tag{3.4}$$

Using the relations (3.2) we find

$$dx = -xy\theta^1, \qquad dy = xy\theta^2 \tag{3.5}$$

and solving for the θ^a we obtain

$$\theta^1 = -q^{-1} x^{-1} y^{-1} dx, \qquad \theta^2 = q^{-1} x^{-1} y^{-1} dy. \tag{3.6}$$

The θ^a satisfy the commutation relations

$$(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad \theta^1\theta^2 + q\theta^2\theta^1 = 0. \tag{3.7}$$

This is of the form (2.11). If we reorder the indices $(11, 12, 21, 22) = (1, 2, 3, 4)$ then the matrix $C^{ab}{}_{cd}$ can be written in the form of a 4×4 matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

That is, $C^{12}{}_{21} = q$ and $C^{21}{}_{12} = q^{-1}$. The structure elements $C^a{}_{bc}$ are given by

$$C^1{}_{12} = -x, \qquad C^2{}_{12} = -y \tag{3.9}$$

and the condition (2.14). Equation (2.18) is satisfied. For θ we find the expression

$$\theta = \frac{1}{q-1}(qx^{-1}dx - y^{-1}dy). \tag{3.10}$$

It is a closed form.

As a second example with $n = 2$ we define, for $q^4 \neq 1$

$$\lambda_1 = \frac{1}{q^4-1}x^{-2}y^2, \qquad \lambda_2 = \frac{1}{q^4-1}x^{-2}. \tag{3.11}$$

The normalization has been chosen here again so that the structure elements $C^a{}_{12}$ contain no factors q. We find then that for $q^2 \neq -1$

$$\begin{aligned} e_1x &= -\frac{1}{q^2(q^2+1)}x^{-1}y^2, & e_1y &= -\frac{1}{q^2+1}x^{-2}y^3, \\ e_2x &= 0, & e_2y &= -\frac{1}{q^2+1}x^{-2}y. \end{aligned} \tag{3.12}$$

From these we conclude that the commutation relations which follow from (2.10) are

$$\begin{aligned} xdx &= q^2dxx, & xdy &= qdyx + (q^2-1)dxy, \\ ydx &= qdxy, & ydy &= q^2dyy. \end{aligned} \tag{3.13}$$

We have then in this case the covariant differential calculus of Wess & Zumino (1990). It has been encoded in the functional form of the λ_a. If $q^2 \neq -1$ from (3.13) we deduce

$$(dx)^2 = 0, \qquad (dy)^2 = 0, \qquad dydx + qdxdy = 0. \tag{3.14}$$

Using the relation (2.6) we find

$$dx = -\frac{1}{q^2(q^2+1)}x^{-1}y^2\theta^1, \qquad dy = -\frac{1}{q^2+1}x^{-2}y(y^2\theta^1 + \theta^2) \tag{3.15}$$

and solving for the θ^a we obtain

$$\theta^1 = -q^4(q^2+1)xy^{-2}dx, \qquad \theta^2 = -q^2(q^2+1)x(xy^{-1}dy - dx). \tag{3.16}$$

The θ^a satisfy the commutation relations

$$(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad q^4\theta^1\theta^2 + \theta^2\theta^1 = 0. \tag{3.17}$$

This is of the form (2.11) if the matrix $C^{ab}{}_{cd}$ is given by the 4×4 matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q^{-4} & 0 \\ 0 & q^4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.18}$$

That is, $C^{12}{}_{21} = q^{-4}$ and $C^{21}{}_{12} = q^4$. The structure elements $C^a{}_{bc}$ are given by

$$C^1{}_{12} = -x^{-2}, \qquad C^2{}_{12} = -x^{-2}y^2 \tag{3.19}$$

and the condition (2.14). Equation (2.18) is again satisfied.

For θ we find the expression

$$\theta = \frac{q^2}{q^2-1} y^{-1} dy. \tag{3.20}$$

It is again a closed form.

From these two examples we see that each choice of two elements λ_1 and λ_2 gives rise to a differential calculus on the generalized quantum plane and we shall see in the Section 5 that each choice gives rise to a linear connection.

4 Calculi based on 3 derivations

In this section we shall consider the case $n = 3$. There is an essential difference with the previous case in that relations of the form (3.3) or (3.13) which allow one to pass from one side of the differential to the other no longer hold. The difference is given in fact in terms of the extra elements of the frame. What we do is extend the definition of dx and dy to another derivation and the extension satisfies quite naturally less relations. The left (or right) module $\Omega^1(\mathcal{A})$ is now of rank 3 instead of 2. As an example we extend the λ_a defined in (3.1) by the addition of a λ_3:

$$\lambda_1 = \frac{q}{q-1} y, \qquad \lambda_2 = \frac{q}{q-1} x, \qquad \lambda_3 = \frac{q}{q-1} \alpha x y. \tag{4.1}$$

The α is an arbitrary complex number. We have then $[\lambda_1, \lambda_2] = -\alpha^{-1}\lambda_3$ but of course the set of λ_a do not form a Lie algebra. To the relations (3.2) we must add two additional ones,

$$e_3 x = -\alpha x^2 y, \qquad e_3 y = \alpha x y^2, \tag{4.2}$$

and so we find

$$dx = -xy\theta^1 - \alpha x^2 y \theta^3, \qquad dy = xy\theta^2 + \alpha x y^2 \theta^3. \tag{4.3}$$

instead of (3.5). Define

$$\tau = x dy - q dy x. \tag{4.4}$$

Then one of the commutation relations (3.3) becomes an expression for θ^3 in terms of τ:

$$\tau = \alpha q^{-1}(q-1) x^2 y^2 \theta^3. \tag{4.5}$$

We can solve then (4.3) for the θ^a and we obtain

$$\begin{aligned}
\theta^1 &= -q^{-1}x^{-1}y^{-1}dx - \frac{1}{q^2(q-1)} x^{-1}y^{-2}\tau, \\
\theta^2 &= q^{-1}x^{-1}y^{-1}dy - \frac{1}{q(q-1)} x^{-2}y^{-1}\tau, \\
\theta^3 &= \frac{1}{\alpha q^3(q-1)} x^{-2}y^{-2}\tau
\end{aligned} \tag{4.6}$$

instead of (3.6). This frame is singular in the limit $q \to 1$ as it must be. The differential calculus, expressed in terms of dx and dy, has however a well-defined limit which lies somewhere between the de Rham calculus and the universal one. For a discussion of this point we refer to Dimakis & Müller-Hoissen (1992), Dimakis & Tzanakis (1995) and to Baehr *et al.* (1995).

If $q \neq -1$ we can deduce the first two of the relations (3.4) and we can conclude that

$$\begin{aligned} q(\theta^1)^2 + \alpha x(\theta^1\theta^3 + q\theta^3\theta^1) + \alpha^2 x^2 (\theta^3)^2 = 0, \\ q(\theta^2)^2 + \alpha y(\theta^3\theta^2 + q\theta^2\theta^3) + \alpha^2 y^2 (\theta^3)^2 = 0. \end{aligned} \tag{4.7}$$

Multiply the first equation by y and the second by x and commute through. One deduces then that each of the coefficients must vanish:

$$(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad (\theta^3)^2 = 0,$$

and

$$\theta^1\theta^3 + q\theta^3\theta^1 = 0, \qquad \theta^3\theta^2 + q\theta^2\theta^3 = 0. \tag{4.8}$$

There is missing a relation between $\theta^1\theta^2$ and $\theta^2\theta^1$. We must therefore rather artificially complete the coefficients in (2.11) by setting $C^{12}{}_{12} = -1$ and $C^{12}{}_{21} = 0$. From (2.23) we find then that $K_{ab} = 0$ and the $D^a{}_{bc}$ vanish except for the values

$$D^3{}_{12} = \frac{2}{\alpha(q-1)}, \qquad D^3{}_{21} = qD^3{}_{12}. \tag{4.9}$$

The form θ is given by

$$\theta = -\frac{q}{q-1}(y\theta^1 + x\theta^2 + \alpha xy\theta^3).$$

It follows then immediatedly from (2.19) that

$$\begin{aligned} d\theta^1 &= \frac{q}{q-1} x(\theta^1\theta^2 + \theta^2\theta^1) + \alpha xy\theta^1\theta^3, \\ d\theta^2 &= \frac{q}{q-1} y(\theta^1\theta^2 + \theta^2\theta^1) + \alpha xy\theta^3\theta^2, \\ d\theta^3 &= y\theta^1\theta^3 + x\theta^3\theta^2 - \frac{1}{\alpha(q-1)}(\theta^1\theta^2 + q\theta^2\theta^1). \end{aligned} \tag{4.10}$$

from which one can read off the expressions for the structure elements which extend (3.9). The third of the relations (3.4) becomes

$$d\tau = -x^2y^2\Big((\theta^1\theta^2 + q\theta^2\theta^1) + \alpha x(\theta^2\theta^3 + \theta^3\theta^2) + \alpha y(\theta^3\theta^1 + \theta^1\theta^3)\Big).$$

Using (4.10) one finds

$$d\tau = -x^2y^2\Big((\theta^1\theta^2 + q\theta^2\theta^1) - \alpha q^{-1}(q-1)d\theta^3\Big). \tag{4.11}$$

If one adds to (4.8) the supplementary relation

$$\theta^1\theta^2 + q\theta^2\theta^1 = 0 \tag{4.12}$$

then $\Omega^2(\mathcal{A})$ becomes a quotient of the right-hand side of (2.4). We can set $C^{12}{}_{21} = q$ and $C^{12}{}_{12} = 0$ as in (3.8). Now we have $K_{ab} = 0$ and $D^a{}_{bc} = 0$ and Equations (4.10) reduce to

$$\begin{aligned} d\theta^1 &= x\theta^1\theta^2 + \alpha xy\theta^1\theta^3, \\ d\theta^2 &= y\theta^1\theta^2 + \alpha xy\theta^3\theta^2, \\ d\theta^3 &= y\theta^1\theta^3 + x\theta^3\theta^2. \end{aligned}$$

Equation (4.11) simplifies to

$$d\tau = \alpha x^2 y^2 q^{-1}(q-1)d\theta^3. \tag{4.13}$$

A similar extension of the second example of the previous section can be given, again by introducing a third derivation. As before this yields an extension of the module of forms as a left (or right) module.

5 Linear connections

For each of the differential calculi defined in the previous section one can define a set of linear connections. The definition of a connection as a covariant derivative was given an algebraic form in the Tata lectures by Koszul (1960) and generalized to noncommutative geometry by Karoubi (1981) and Connes (1986, 1994). We shall often use here the expressions 'connection' and 'covariant derivative' synonymously. In fact we shall distinguish three different types of connections. A 'left $\mathcal{A}$-connection' is a connection on a left $\mathcal{A}$-module; it satisfies a left Leibniz rule. A 'bimodule $\mathcal{A}$-connection' is a connection on a general bimodule $\mathcal{M}$ which satisfies a left and right Leibniz rule. In the particular case where $\mathcal{M}$ is the module of 1-forms we shall speak of a 'linear connection'. A bimodule over an algebra $\mathcal{A}$ is also a left module over the tensor product $\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{\mathrm{op}}$ of the algebra with its 'opposite'. Such a bimodule can have a bimodule $\mathcal{A}$-connection as well as a left $\mathcal{A}^e$-connection. (Cuntz & Quillen 1995, Bresser *et al.* 1995). These two definitions are compared in Dubois-Violette *et al.* (1995b).

Let $\mathcal{A}$ be an arbitrary algebra and $(\Omega^*(\mathcal{A}), d)$ a differential calculus over $\mathcal{A}$. One defines a left $\mathcal{A}$-connection on a left $\mathcal{A}$-module $\mathcal{H}$ as a covariant derivative

$$\mathcal{H} \stackrel{D}{\rightarrow} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \tag{5.1}$$

which satisfies the left Leibniz rule

$$D(f\psi) = df \otimes \psi + fD\psi \tag{5.2}$$

for arbitrary $f \in \mathcal{A}$. This map has a natural extension

$$\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \stackrel{\nabla}{\longrightarrow} \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \tag{5.3}$$

given, for $\psi \in \mathcal{H}$ and $\alpha \in \Omega^n(\mathcal{A})$, by $\nabla\psi = D\psi$ and

$$\nabla(\alpha\psi) = d\alpha \otimes \psi + (-1)^n \alpha \nabla\psi.$$

The operator ∇^2 is necessarily left-linear. However when $\mathcal{H}$ is a bimodule it is not in general right-linear.

A covariant derivative on the module $\Omega^1(\mathcal{A})$ must satisfy (5.2). But $\Omega^1(\mathcal{A})$ has also a natural structure as a right $\mathcal{A}$-module and one must be able to write a corresponding right Leibniz rule in order to construct a bilinear curvature. Quite generally let $\mathcal{M}$ be an arbitrary bimodule. Consider a covariant derivative

$$\mathcal{M} \stackrel{D}{\to} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M} \tag{5.4}$$

which satisfies both a left and a right Leibniz rule. In order to define a right Leibniz rule which is consistent with the left one, it was proposed by Mourad (1995), by Dubois-Violette & Michor (1995) and by Dubois-Violette & Masson (1995) to introduce a generalized permutation

$$\mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \stackrel{\sigma}{\longrightarrow} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}.$$

The right Leibniz rule is given then as

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f \tag{5.5}$$

for arbitrary $f \in \mathcal{A}$ and $\xi \in \mathcal{M}$. The purpose of the map σ is to bring the differential to the left while respecting the order of the factors. It is necessarily bilinear (Dubois-Violette *et al.* 1995a). Let π be the product in the algebra of forms. It was argued by Mourad (1995) and by Dubois-Violette *et al.* (1995a) that a necessary as well as sufficient condition for torsion to be right-linear is that σ satisfy the consistency condition

$$\pi \circ (\sigma + 1) = 0. \tag{5.6}$$

We define a bimodule $\mathcal{A}$-connection to be the couple (D, σ). We shall make no mention of curvature. There is at the moment no general concensus of the correct definition of the curvature of a bimodule connection. The problem is that the operator ∇^2 is not in general right-linear. We refer to Dubois-Violette *et al.* (1995b) for a recent discussion.

This general formalism can be applied in particular to the differential calculi which we have constructed in Section 2. Since $\Omega^1(\mathcal{A})$ is a free module the map σ can be defined by the action on the basis elements:

$$\sigma(\theta^a \otimes \theta^b) = S^{ab}{}_{cd}\theta^c \otimes \theta^d. \tag{5.7}$$

By the sequence of identities

$$f S^{ab}{}_{cd}\theta^c \otimes \theta^d = \sigma(f\theta^a \otimes \theta^b) = \sigma(\theta^a \otimes \theta^b f) = S^{ab}{}_{cd} f\theta^c \otimes \theta^d$$

we conclude that the coefficients $S^{ab}{}_{cd}$ must lie in $\mathcal{Z}(\mathcal{A})$. From (2.12) we see that the condition (5.6) can be written

$$(\delta^a_e \delta^b_f + S^{ab}{}_{ef})(\delta^e_c \delta^f_d - C^{ef}{}_{cd}) = 0. \tag{5.8}$$

A natural, but certainly not the unique, choice of σ is given by $S^{ab}{}_{cd} = C^{ab}{}_{cd}$.

Since $\Omega^1(\mathcal{A})$ is a free module a covariant derivative can be defined by its action on the basis elements:

$$D\theta^a = -\omega^a{}_{bc}\theta^b \otimes \theta^c. \tag{5.9}$$

The coefficients here are elements of the algebra. Because of the identity $D(f\theta^a) = D(\theta^a f)$ there are very stringent compatibility conditions, which using (5.7) can be written out as

$$[\omega^a{}_{bc}, f] = e_d f (S^{ad}{}_{bc} - \delta^d_b \delta^a_c).$$

The general solution to this equation is of the form $\omega^a{}_{bc} = \omega_{(0)}{}^a{}_{bc} + \chi^a{}_{bc}$ where

$$\omega_{(0)}{}^a{}_{bc} = \lambda_d (S^{ad}{}_{bc} - \delta^d_b \delta^a_c) \tag{5.10}$$

and $\chi^a{}_{bc} \in \mathcal{Z}(\mathcal{A})$. One can also express $D_{(0)}$ in the form (Dubois-Violette *et al.* 1995a, Madore *et al.* 1995)

$$D_{(0)}\theta^a = -\theta \otimes \theta^a + \sigma(\theta^a \otimes \theta).$$

The torsion 2-form is defined as usual as

$$\Theta^a = d\theta^a - \pi \circ D\theta^a$$

Comparing (5.10) with (2.19), we see that the torsion $\Theta^a_{(0)}$ of $D_{(0)}$ is given by

$$\Theta^a_{(0)} = -\frac{1}{2} D^a{}_{bc}\theta^b\theta^c. \tag{5.11}$$

In general a covariant derivative is torsion-free provided the condition

$$\omega^a{}_{bc} - \omega^a{}_{de} C^{de}{}_{bc} = C^a{}_{bc}$$

is satisfied. The covariant derivative (5.9) is torsion free if and only if

$$\chi^a{}_{bc} = \frac{1}{2} D^a{}_{bc}.$$

On the ordinary quantum plane one can show that there is there is a unique 1-parameter family of linear connections (Dubois-Violette *et al.* 1995a) and that this connection is torsion free. We find here a different result; there is an ambiguity which depends on elements of $\mathcal{Z}(\mathcal{A})$. An interesting limit case is given by

$$S^{ab}{}_{cd} = C^{ab}{}_{cd} = \delta^b_c \delta^a_d. \tag{5.12}$$

In this case from (2.18) one sees that $D^a{}_{bc} = C^a{}_{bc} \neq 0$. From (2.22) one sees that $K_{ab} = 0$ and the λ_a form a Lie algebra. The matrix case is a typical example. From (5.10) if follows that $D_{(0)}\theta^a = 0$ and so $D_{(0)}$ has torsion but no curvature. The connection corresponds to the canonical flat connection on a parallelizable manifold.

One can define a metric by the condition

$$g(\theta^a \otimes \theta^b) = g^{ab} \tag{5.13}$$

where the coefficients g^{ab} are elements of the algebra. To be well defined on all elements of the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ the metric must be bilinear and by the sequence of identities

$$f g^{ab} = g(f\theta^a \otimes \theta^b) = g(\theta^a \otimes \theta^b f) = g^{ab} f \tag{5.14}$$

we conclude that the coefficients must lie in $\mathcal{Z}(\mathcal{A})$. The covariant derivative (5.9) is compatible with the metric (Dubois-Violette *et al.* 1995a) if and only if

$$\omega^a{}_{bc} + \omega_{ce}{}^f S^{ae}{}_{bf} = 0. \tag{5.15}$$

The condition that (5.10) be metric compatible can be written as

$$S^{ae}{}_{dh} g^{hf} S^{cb}{}_{ef} = g^{ac}\delta^b_d. \tag{5.16}$$

Consider now the first differential calculus of Section 3, defined by (3.1). On the right-hand side of (2.23) we have $K_{ab} = 0$ and $D^a{}_{bc} = 0$. The torsion of $D_{(0)}$ vanishes. The coefficients g^{ab} are complex numbers. With the convention of (3.8) they can be written as (g^1, g^2, g^3, g^4). Using the $GL(2, \mathbb{C})$-invariance one can impose that

$$g^4 = g^1, \qquad g^3 = -g^2.$$

If we suppose that $g^2 = 0$ there is no restriction in supposing that $g^1 = 1$; the g^{ab} are the components of the euclidean metric in two dimensions. With the convention of (3.8) the condition (5.16) can be written in the matrix form

$$\begin{pmatrix} S^1{}_1 & S^1{}_2 & S^1{}_3 & S^1{}_4 \\ S^2{}_1 & S^2{}_2 & S^2{}_3 & S^2{}_4 \\ S^3{}_1 & S^3{}_2 & S^3{}_3 & S^3{}_4 \\ S^4{}_1 & S^4{}_2 & S^4{}_3 & S^4{}_4 \end{pmatrix} \begin{pmatrix} S^1{}_1 & S^1{}_3 & S^3{}_1 & S^3{}_3 \\ S^1{}_2 & S^1{}_4 & S^3{}_2 & S^3{}_4 \\ S^2{}_1 & S^2{}_3 & S^4{}_1 & S^4{}_3 \\ S^2{}_2 & S^2{}_4 & S^4{}_2 & S^4{}_4 \end{pmatrix} = 1. \tag{5.17}$$

From the approximation linear in $q - 1$ one sees that the solution must be of the form

$$S = \begin{pmatrix} S^1{}_1 & 0 & 0 & S^1{}_4 \\ 0 & S^2{}_2 & S^2{}_3 & 0 \\ 0 & S^3{}_2 & S^3{}_3 & 0 \\ S^4{}_1 & 0 & 0 & S^4{}_4 \end{pmatrix}. \tag{5.18}$$

The consistancy conditions (5.8) become

$$1 + S^2{}_2 = q^{-1} S^2{}_3, \qquad 1 + S^3{}_3 = q S^3{}_2. \tag{5.19}$$

In general $S^{ab}{}_{cd} = C^{ab}{}_{cd}$ does not yield a metric-compatible covariant derivative. There is a solution however to (5.17), (5.19) given by

$$S = \frac{1}{q^2 + 1} \begin{pmatrix} 2q & 0 & 0 & 1 - q^2 \\ 0 & 1 - q^2 & 2q & 0 \\ 0 & 2q & q^2 - 1 & 0 \\ q^2 - 1 & 0 & 0 & 2q \end{pmatrix} \tag{5.20}$$

That is, for example,

$$S^{12}{}_{21} = S^{21}{}_{12} = \frac{2q}{q^2 + 1}.$$

The expression (5.20) has the same limit as (3.8) when $q \to 1$, as it must for the right-hand side of (5.10) to exist. With σ given by (5.7), the covariant derivative is compatible with the metric (5.13) and torsion free. Comparing (3.18) with (3.8) one sees that one obtains for the second example (3.11) a covariant derivative compatible with the metric (5.13) by the replacement $q \mapsto q^{-4}$ in (5.20). The dependence on q comes through the conditions (5.19). Since $S(q) = -S(-q^{-1})$ the same matrix serves two different values of the parameter q.

The metric we have chosen is not symmetric with respect to σ. That is

$$g^{ab} \neq S^{ab}{}_{cd} g^{cd}$$

in general. If one wishes to find a metric symmetric in the above sense then one must consider (5.16) as an equation for S and the metric and add the additional equation

$$g^{ab} = S^{ab}{}_{cd} g^{cd}. \tag{5.21}$$

The system (5.16), (5.21), without the restricton we have placed on the coefficients g^{ab}, if it has a solution, would yield a symmetric metric with a compatible connection.

Restricting one's attention to hermitian x and y and real q, in the limit $q \to 1$ one obtains on the ordinary 2-plane a metric whose Gaussian curvature K is given by

$$K_1 = x^2 + y^2, \qquad K_2 = x^{-4}(1 + y^4) \tag{5.22}$$

respectively for the two examples of Section 3. This can be calculated using the $q \to 1$ limit of (5.10). It is easy to characterize all metrics which can be obtained in this way. In the limit $q \to 1$ the commutator determines a Poisson bracket on the 2-plane given as usual by

$$\{f, g\} = \lim_{q \to 1} \frac{1}{q-1}[f, g]. \tag{5.23}$$

Define

$$p_a = \lim_{q \to 1}(q-1)\lambda_a.$$

In the limit the differential can be written then in the form

$$df = \{p_a, f\}\theta^a. \tag{5.24}$$

If we write $\theta^a = \theta^a_b dx^b$ in terms of dx^a from this it follows that the equation

$$\{p_c, x^a\}\theta^c_b = \delta^a_b. \tag{5.25}$$

must have a solution for p_a polynomial in the variables. This is not always the case. That is, not all metrics with polynomial curvature can be obtained as were those given by (5.22). For example consider the flat metric $\theta^a_b = \delta^a_b$. The equations (5.25) become the equations $\{p_a, x^b\} = \delta^a_b$. Using the expression for the Poisson bracket for the generalized quantum plane, $\{x, y\} = xy$, one sees immediately that there is no solution for the p_a.

The generalized quantum plane has two outer derivations defined by

$$e^{(0)}_1 x = x, \qquad e^{(0)}_1 y = 0, \qquad e^{(0)}_2 x = 0, \qquad e^{(0)}_2 y = y. \tag{5.26}$$

The corresponding θ^a are given by

$$\theta^1 = x^{-1}dx, \qquad \theta^2 = y^{-1}dy. \tag{5.27}$$

Our construction yields then the ordinary flat metric. If one were to extend the algebra to the Heisenberg algebra then these derivations would become interior. To obtain a metric which is almost flat one can add to (5.26) a 'small' inner derivation of the form given in Section 2 but using λ_a which are 'small' of the order of some expansion parameter ϵ. One defines

$$e_a = e^{(0)}_a + \epsilon \,\mathrm{ad}\, \lambda_a \tag{5.28}$$

and proceeds as above but retaining only corrections of first order in ϵ. A problem closely related to this has been examined in another context by one of the authors (Madore 1995b).

The equations (5.17), (5.19) admit also the solution

$$S = \frac{1}{q^2+1}\begin{pmatrix} -2q & 0 & 0 & 1-q^2 \\ 0 & 1-q^2 & 2q & 0 \\ 0 & 2q & q^2-1 & 0 \\ q^2-1 & 0 & 0 & -2q \end{pmatrix} \tag{5.29}$$

but the corresponding covariant derivative defined by (5.10) is singular in the limit $q \to 1$.

A complete study of the matrix case has not been made. However for the particular case (5.12) it is easy to see that the unique torsion-free covariant derivative compatible with the metric (5.13) is given by

$$D\theta^a = -\frac{1}{2} C^a{}_{bc} \theta^b \otimes \theta^c. \tag{5.30}$$

The ordinary quantum plane with the differential calculus given by (3.13) has no metric connection but it posesses a unique 1-parameter family of linear connections whose curvature is polynomial in the variables x and y (Dubois-Violette *et al.* 1995a). The precise property of the curvature K_2 in (5.22) which associates the corresponding metric to the $GL_q(2)$-invariant differential calculus (3.13) is not clear. We refer to Madore & Mourad (1996) for a description of the possible relevance to the theory of gravity of the relation between linear connections on the one hand and commutation relations on the other.

6 Conclusions

We have shown that each differential calculus and set of commutation relations between the 1-forms and the elements of the algebra gives rise in the case of the generalized quantum plane to a metric connection which remains regular in the limit $q \to 1$. Not all metrics with polynomial curvature can be obtained in this way.

7 Acknowledgment

One of the authors (J.M.) would like to thank M. Dubois-Violette, T. Masson and J. Mourad for interesting conversations.

References

Baehr H.C., Dimakis A., Müller-Hoissen F. 1995, *Differential calculi on commutative algebras*, J. Phys. A **28** 3197.

Bresser K., Müller-Hoissen F., Dimakis A., Sitarz A. 1995, *Noncommutative Geometry of Finite Groups*, Preprint GOET-TP 95/95.

Connes A. 1986, *Non-Commutative Differential Geometry*, Publications of the Inst. des Hautes Etudes Scientifique. **62** 257.

— 1994, *Noncommutative Geometry*, Academic Press.

Cuntz A., Quillen D. 1995, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** 251.

Dimakis A., Müller-Hoissen F. 1992, *Noncommutative differential calculus, gauge theory and gravitation*, Preprint GOE-TP 33/92.

Dimakis A., Tzanakis C. 1995, *Noncommutative Geometry and Kinetic Theory of Open systems*, hep-th/9508035, J. Phys. A. (to appear).

Dubois-Violette M. 1988, *Dérivations et calcul différentiel non-commutatif*, C. R. Acad. Sci. Paris **307** Série I 403.

Dubois-Violette M., Kerner R., Madore J. 1989, *Gauge bosons in a noncommutative geometry*, Phys. Lett. **B217** 485; *Classical bosons in a noncommutative geometry*, Class. Quant. Grav. **6** 1709.

Dubois-Violette M., Madore J., Masson T., Mourad J. 1995a, *Linear Connections on the Quantum Plane*, Lett. Math. Phys. **35** 351.

— 1995b, *On Curvature in Noncommutative Geometry*, Preprint LPTHE Orsay 95/63; q-alg/9512004.

Dubois-Violette M., Michor P. 1995, *Connections on Central Bimodules*, Preprint LPTHE Orsay 94/100.

Dubois-Violette M., Masson T. 1995, *On the First-Order Operators in Bimodules*, Preprint LPTHE Orsay 95/56.

Karoubi M. 1981, *Connections, courbures et classes caractéristiques en K-théorie algébrique.*, Current trends in algebraic topology, Part I, London, Ont.

Koszul J.L. 1960, *Lectures on Fibre Bundles and Differential Geometry*, Tata Institute of Fundamental Research, Bombay.

Madore J. 1995a, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press.

— 1995b, *Linear Connections on Fuzzy Manifolds*, Preprint LPTHE Orsay 95/42, ESI Vienna 235, hep-th/9506183.

Madore J., Mourad. J. 1996, *Noncommutative Kaluza-Klein Theory*, Lecture given at the 5$^{\text{th}}$ Hellenic School and Workshops on Elementary Particle Physics (to be published).

Madore J., Masson T., Mourad J. 1995, *Linear Connections on Matrix Geometries*, Class. Quant. Grav. **12** 1429.

Mourad. J. 1995, *Linear Connections in Non-Commutative Geometry*, Class. Quant. Grav. **12** 965.

Wess J., Zumino B. 1990, *Covariant Differential Calculus on the Quantum Hyperplane* Nucl. Phys. B (Proc. Suppl.) **18B** 302.